51. Für A: $\lambda = -1$:

$$\begin{aligned} 2x &= -x \\ -y + 2z &= -y \\ -z &= -z \end{aligned} \iff x = 0, \; z = 0$$

$\Rightarrow \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ ist eine Basis von $E(-1)$.

$\lambda = 2$:

$$\begin{aligned} 2x &= 2 \\ -y + 2z &= 2y \\ -z &= 2z \end{aligned} \iff z = 0, \; y = 0$$

$\Rightarrow \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ ist ein Basis von $E(2)$.

Für B: $\lambda = -1$:

$$\begin{aligned} -5x \quad + 7z &= -x \\ 6x + 2y - 6z &= -y \\ -4x \quad + 6z &= -z \end{aligned} \iff \begin{aligned} -4x \quad + 7z &= 0 \\ 6x + 3y - 6z &= 0 \\ -4x \quad + 7z &= 0 \end{aligned}$$

$$\underset{\substack{I \cdot 6 \\ II \cdot 4}}{\iff} \begin{aligned} -24x \quad + 42z &= 0 \\ 24x + 12y - 24z &= 0 \end{aligned} \underset{I + II}{\iff} \begin{aligned} -24x \quad + 42z &= 0 \\ 12y + 18z &= 0 \end{aligned}$$

$$\underset{\substack{I/6 \\ II/6}}{\iff} \begin{aligned} -4x \quad + 7z &= 0 \\ 2y + 3z &= 0 \end{aligned} \iff y = -\frac{3}{2} z, \; x = \frac{7}{4} z$$

$\Rightarrow \left(\begin{pmatrix} 7 \\ -6 \\ 4 \end{pmatrix}\right)$ ist eine Basis von $E(-1)$.

$\underline{\lambda = 2}$:

$$\begin{array}{lllll} -5x & & +7z & = 2x & \iff & -7x & & +7z & = 0 & \iff x = z \\ 6x & +2y & -6z & = 2y & & 6x & & -6z & = 0 \\ -4x & & +6z & = 2z & & -4x & & +4z & = 0 \end{array}$$

$\Rightarrow \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$ ist eine Basis von $E(2)$.

52. $\underline{\lambda = 0}$:

$$\begin{array}{rrrrl} & 2b & +2c & -d & =0 \\ -2a & +3b & & & =0 \\ -2a & & -3c & +d & =0 \\ & & & d & =0 \end{array} \iff \begin{array}{rrrrl} & b & +c & & =0 \\ -2a & +3b & & & =0 \\ -2a & & -3c & & =0 \\ & & & d & =0 \end{array}$$

$$\underset{\text{II}-\text{III}}{\iff} \begin{array}{rrrrl} & b & +c & & =0 \\ & 3b & +3c & & =0 \\ -2a & & -3c & & =0 \\ & & & d & =0 \end{array} \iff d=0,\ b=-c,\ a=-\tfrac{3}{2}c$$

$$\Rightarrow \left(\begin{pmatrix} -3 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right) \text{ ist eine Basis von } E(0)$$

$\underline{\lambda = 1}$:

$$\begin{array}{rrrrl} & 2b & +2c & -d & =a \\ -2a & +3b & & & =b \\ -2a & & -3c & +d & =c \\ & & & d & =d \end{array} \iff \begin{array}{rrrrl} -a & +2b & +2c & -d & =0 \\ -2a & +2b & & & =0 \\ -2a & & -4c & +d & =0 \end{array}$$

$$\underset{\text{I}\cdot 2}{\iff} \begin{array}{rrrrl} -2a & +4b & +4c & -2d & =0 \\ -2a & +2b & & & =0 \\ -2a & & -4c & +d & =0 \end{array} \underset{\substack{\text{II}-\text{I} \\ \text{III}-\text{I}}}{\iff} \begin{array}{rrrrl} -2a & +4b & +4c & -2d & =0 \\ & -2b & -4c & +2d & =0 \\ & -4b & -8c & +3d & =0 \end{array}$$

$$\underset{\substack{\text{I}/2 \\ \text{II}/2 \\ \text{III}-2\cdot\text{II}}}{\iff} \begin{array}{rrrrl} -a & +2b & +2c & -d & =0 \\ & -b & -2c & +d & =0 \\ & & & -d & =0 \end{array} \iff \begin{array}{rrrrl} -a & +2b & +2c & & =0 \\ & -b & -2c & & =0 \\ & & & d & =0 \end{array}$$

$$\underset{\text{I}+2\cdot\text{II}}{\iff} \begin{array}{rrrl} -a & & -2c & =0 \\ & -b & -2c & =0 \\ & & d & =0 \end{array} \iff d=0,\ b=-2c,\ a=-2c$$

$$\Rightarrow \left(\begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right) \text{ ist eine Basis von } E(1).$$

$\underline{\lambda = -1:}$

$$\begin{aligned} 2b + 2c - d &= -a \\ -2a + 3b &= -b \\ -2a - 3c + d &= -c \\ d &= -d \end{aligned} \iff \begin{aligned} a + 2b + 2c &= 0 \\ -2a + 4b &= 0 \\ -2a - 2c &= 0 \\ d &= 0 \end{aligned}$$

$$\iff \begin{array}{l} \\ II/2 \\ III/2 \end{array} \begin{aligned} a + 2b + 2c &= 0 \\ -a + 2b &= 0 \\ -a - c &= 0 \\ d &= 0 \end{aligned} \iff \begin{array}{l} \\ II+I \\ III+I \end{array} \begin{aligned} a + 2b + 2c &= 0 \\ 4b + 2c &= 0 \\ 2b + c &= 0 \\ d &= 0 \end{aligned}$$

$$\iff \begin{aligned} a + 2b + 2c &= 0 \\ 2b + c &= 0 \\ d &= 0 \end{aligned} \iff d = 0,\ b = -\tfrac{1}{2}c,\ a = -c.$$

$$\Rightarrow \left(\begin{pmatrix} -2 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right) \text{ ist eine Basis von } E(-1)$$

Die geometrische Vielfachheiten von 0, 1 und −1 ist 1.
$\mathbb{R}^4$ hat kein Basis aus Eigenvektoren, also ist C nicht diagonalisierbar.

53. Der Definitionsbereich von f und F ist $\mathbb{R}\setminus\{-3,4\}$ in (a), $(-1,1)$ in (b) und $\mathbb{R}$ in (c) – (f).

(a) $(-x^2+x-2):(x^2-x-12) = -1$ mit Rest -14,

$$\frac{\begin{array}{l}-x^2+x+12\end{array}}{-14}$$

$$\frac{-x^2+x-2}{x^2-x-12} = 1 - \frac{14}{x^2-x-12}.$$

Die Nullstellen von x^2-x-12 sind $-3,4$ und

$$x^2-x-12 = (x+3)(x-4).$$

Für $r(x) = 14$, $q(x) = (x+3)(x-4)$ ist

$r(x) = 14 = -2(x+3) + 2(x-4)$ (durch Lösen eines linearen Gleichungssystems),

$$\frac{r(x)}{q(x)} = -\frac{2}{x-4} + \frac{2}{x+3}.$$

$$\int \frac{-x^2+x-2}{x^2-x-12}\,dx = \int -1\,dx + \int -\frac{2}{x-4}\,dx + \int \frac{2}{x+3}\,dx =$$

$$= -x - 2\log|x-4| + 2\log|x+3|.$$

(b) $x\operatorname{Log}(1-x^2) = g'(x)\,h(x)$ für $g(x) = \frac{1}{2}x^2$, $h(x) = \log(1-x^2)$.

$$h'(x) = \frac{1}{1-x^2}(-2x) = -\frac{2x}{1-x^2}.$$

$$g(x)h'(x) = -\frac{x^3}{1-x^2} = \frac{x^3}{x^2-1}.$$

$x^3:(x^2-1) = x$ mit Rest x, also ist $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$.

$$\frac{\begin{array}{l}x^3-x\end{array}}{x}$$

Für $r(x) = x$, $q(x) = x^2-1$ ist $r(x) = x = \frac{1}{2}(x-1) + \frac{1}{2}(x+1)$

und $\frac{r(x)}{q(x)} = \frac{x}{x^2-1} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)}$.

$$\int \frac{x^3}{x^2-1} = \int x\,dx + \int \frac{x}{x^2-1}\,dx = \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx =$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\log|x+1| + \frac{1}{2}\log|x-1|$$

$$\int x\log(1-x^2)\,dx = \frac{1}{2}x^2\operatorname{Log}(1-x^2) - \frac{1}{2}x^2 - \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x-1|.$$

(c) $x e^x = g'(x) h(x)$ für $g(x) = e^x$, $h(x) = x$.

$g(x) h'(x) = e^x$

$\int x e^x dx = x e^x - \int e^x = x e^x - e^x = (x-1) e^x$.

(d) $x^3 e^{-x^2} = -\frac{1}{2} x^2 \cdot (-2x e^{-x^2}) = g'(x) h(x)$ für

$g(x) = e^{-x^2}$, $h(x) = -\frac{1}{2} x^2$.

$$\int x^3 e^{-x^2} dx = -\frac{1}{2} x^2 e^{-x^2} - \int g(x) h'(x) dx$$
$$= -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} dx = \frac{1}{2} x^2 e^{-x^2} + \frac{1}{2} e^{-x^2}$$
$$= \frac{x^2+1}{2} e^{-x^2}.$$

(e) $x^2 \cos(x) = g'(x) h(x)$ für $g(x) = \sin(x)$, $h(x) = x^2$.

$g(x) h'(x) = 2x \sin(x) = u'(x) v(x)$ für $u(x) = -\cos(x)$, $v(x) = 2x$.

$u(x) v'(x) = -2 \cos(x)$.

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \int 2x \sin(x) dx$$

$$\int 2x \sin(x) dx = -2x \cos(x) - \int -2 \cos(x) dx = -2x \cos(x) + 2 \sin(x)$$

$$\Rightarrow \int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) =$$
$$= (x^2 - 2) \sin(x) + 2x \cos(x)$$

(f) $\cos(2x) = \cos(x)^2 - \sin(x)^2 = \cos(x)^2 - (1 - \cos(x)^2) = 2\cos(x)^2 - 1$

$\Rightarrow \cos(x)^2 = \frac{1}{2} (\cos(2x) + 1)$.

$$\int \cos(x)^2 dx = \frac{1}{2} \int \cos(2x) dx + \frac{1}{2} \int 1 \, dx = \frac{1}{4} \sin(2x) + \frac{1}{2} x =$$
$$= \frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x = \frac{1}{2} \sin(x) \sqrt{1 - \sin(x)^2} + \frac{1}{2} x.$$

54. (a) $\operatorname{Ar\,sinh}(\sinh(x)) = \log\left(\sinh(x) + \sqrt{\sinh(x)^2+1}\right)$.

$$\sinh(x) + \sqrt{\sinh(x)^2+1} = \frac{e^x - e^{-x}}{2} + \sqrt{\left(\frac{e^x-e^{-x}}{2}\right)^2+1} =$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x} + \sqrt{\frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} - \frac{1}{2} + 1}.$$

Also ist $\sinh(x) + \sqrt{\sinh(x)^2+1} = e^x \iff$

$$\iff \sqrt{\frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \frac{1}{2}} = \frac{1}{2}e^x + \frac{1}{2}e^{-x} \iff$$

$$\iff \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2 = \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \frac{1}{2}.$$

(b) Wir substituieren $x = g(t) = \sinh(t)$.

$$F(x) = \int \frac{1}{\sqrt{x^2+1}}$$

$$F(g(t)) = F(\sinh(t)) = \int f(g(t))\, g'(t)\, dt = \int \frac{\cosh(t)}{\sqrt{\sinh(t)^2+1}}\, dt = t$$

$$\Rightarrow F(x) = g^{-1}(x) = \operatorname{Ar\,sinh}(x) = \log\left(x + \sqrt{x^2+1}\right)$$

(c) $$\operatorname{Ar\,sinh}'(x) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right) = \frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1} = \frac{1}{\sqrt{x^2+1}}.$$

$$\operatorname{Ar\,sinh}'(x) = \frac{1}{\sinh'(\operatorname{Ar\,sinh}(x))},$$

$$\sinh'(x) = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x),$$

$$\cosh(\operatorname{Ar\,sinh}(x)) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right) + \frac{1}{2\left(x+\sqrt{x^2+1}\right)} =$$

$$= \frac{\left(x+\sqrt{x^2+1}\right)^2+1}{2\left(x+\sqrt{x^2+1}\right)} = \frac{2x^2+2+2x\sqrt{x^2+1}}{2\left(x+\sqrt{x^2+1}\right)} = \frac{x^2+1+x\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \sqrt{x^2+1}.$$

(d) Wir substituieren $x = g(t) = \sin(t)$.

$$F(x) = \int \sqrt{1-x^2}\, dx,$$

$$F(g(t)) = F(\sin(t)) = \int f(g(t))\, g'(t)\, dt = \int \sqrt{1-\sin(t)^2}\, \cos(t)\, dt =$$

$$= \int \cos(t)^2\, dt = \frac{1}{2}\sin(t)\sqrt{1-\sin(t)^2} + \frac{1}{2}t$$ nach Aufgabe 53 (f).

$$\Rightarrow F(x) = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)$$ für $x \in (-1,1)$.

55. (a) $y' = \sqrt{y}$, $y(3) = 4$

In der Notation der Vorlesung sind

$a(x) = \sqrt{x}$ für $x \in \mathbb{R}^+$,

$A(x) = \int \frac{1}{a(x)}\,dx = 2\sqrt{x}$ für $x \in \mathbb{R}^+$,

$A^{-1}(x) = \frac{1}{4}x^2$ für $x \in \mathbb{R}$.

Also löst $y(t) = A^{-1}(t + A(4) - 3) = \frac{1}{4}(t + 2\sqrt{4} - 3)^2 =$

$= \frac{1}{4}(t+1)^2$ das Anfangswertproblem.

(b) Für $y(t) = e^{\lambda t}$ ist

$2y^{(3)} + y'' - 7y' - 6y = (2\lambda^3 + \lambda^2 - 7\lambda - 6)\,e^{\lambda x} = 0$.

Durch Ausprobieren finden wir eine Nullstelle von $2\lambda^3 + \lambda^2 - 7\lambda - 6$: -1.

$$
\begin{array}{l}
(2\lambda^3 + \lambda^2 - 7\lambda - 6) : (\lambda + 1) = 2\lambda^2 - \lambda - 6 \\
\underline{2\lambda^3 + 2\lambda^2} \\
\quad -\lambda^2 - 7\lambda - 6 \\
\quad \underline{-\lambda^2 - \lambda} \\
\qquad -6\lambda - 6
\end{array}
$$

Die Lösungen von $2\lambda^2 - \lambda - 6 = 0$ sind $\frac{1 \pm \sqrt{1+48}}{4} = 2, -\frac{3}{2}$.

Also sind e^{-x}, e^{2x} und $e^{-\frac{3}{2}x}$ Lösungen der Differentialgleichung.

e^{-x}, e^{2x} und $e^{-\frac{3}{2}x}$ sind linear unabhängig.